

Closing Tue: TN 4      Closing Thu: TN 5

### Entry Tasks (Sigma Notation Practice)

1. Differentiate and integrate:

$$f(x) = \sum_{k=2}^5 \frac{(-1)^k}{k^3} x^k = \frac{1}{8}x^2 - \frac{1}{27}x^3 + \frac{1}{64}x^4 - \frac{1}{125}x^5$$

$$f'(x) = \sum_{k=2}^5 \frac{(-1)^k}{k^3} k x^{k-1} = \frac{1}{4}x - \frac{1}{9}x^2 + \frac{1}{16}x^3 - \frac{1}{25}x^4 = \sum_{k=2}^5 \frac{(-1)^k}{k^2} x^{k-1}$$

$$\int f(x) dx = \sum_{k=2}^5 \frac{(-1)^k}{k^3} \frac{1}{(k+1)} x^{k+1} = \frac{1}{8} \frac{1}{3} x^3 - \frac{1}{27} \frac{1}{4} x^4 + \frac{1}{64} \frac{1}{5} x^5 - \frac{1}{125} \frac{1}{6} x^6$$

2. Combine

$$5 \sum_{k=2}^4 k^2 x^k - 6 \sum_{k=2}^4 \frac{1}{k!} x^k = 5(2^2 x^2 + 3^2 x^3 + 4^2 x^4) - 6\left(\frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4\right)$$
$$= \sum_{k=2}^4 \left(5k^2 - \frac{6}{k!}\right) x^k$$

## TN 5: Using Taylor Series

Here are the 6 series you can quote:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \quad \text{for all } x$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \text{for all } x$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \text{for all } x$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{for } -1 < x < 1$$

$$-\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1}, \quad -1 < x < 1$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad -1 < x < 1$$

## Tools for using Taylor Series

1. Substitute (replace x)

2. Integrate

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

3. Differentiate

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

4. Combine

$$\sum_{k=0}^{\infty} kx^k - 3 \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \left( k - \frac{3}{k!} \right) x^k$$

**Substitution Questions:** Find the Taylor series based at 0, find the first three nonzero terms and give the interval of convergence.

$$(a) f(x) = 3e^{2x} = 3 \sum_{k=0}^{\infty} \frac{1}{k!} (2x)^k = \sum_{k=0}^{\infty} \frac{3}{k!} (2)^k x^k = 3 + \frac{3}{1!} 2^1 x^1 + \frac{3}{2!} 2^2 x^2 + \dots$$

$$= 3 + 6x + 6x^2 + \dots$$

For All  $x$   $[-\infty < x < \infty]$

$$(b) g(x) = \frac{5}{1-4x} = 5 \sum_{k=0}^{\infty} (4x)^k = \sum_{k=0}^{\infty} 5(4)^k x^k = 5 + 5(4)^1 x^1 + 5(4)^2 x^2 + \dots$$

$$= 5 + 20x + 80x^2 + \dots$$

for  $-1 < 4x < 1 \Rightarrow [-\frac{1}{4} < x < \frac{1}{4}]$

$$(c) h(x) = \frac{3}{2x+1} = 3 \cdot \frac{1}{1-(-2x)} = 3 \sum_{k=0}^{\infty} (-2x)^k = \sum_{k=0}^{\infty} 3(-2)^k x^k$$

$$= 3 + 3(-2)x + 3(-2)^2 x^2 + \dots$$

$$= 3 - 6x + 12x^2 + \dots$$

for  $-1 < -2x < 1 \Rightarrow [-\frac{1}{2} < x < \frac{1}{2}]$

Combining: Find the Taylor series based at 0, find the first three nonzero terms and give the interval of convergence

$$(a) y = 7 + 3x^5 e^{2x} = 7 + 3x^5 \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = 7 + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} x^{k+5}$$

$$= 7 + 3x^5 + \frac{3(2)}{1!} x^6 + \dots$$

For All  $x$   $-\infty < x < \infty$

$$(b) y = \frac{5}{1-4x} - \frac{3}{2x+1} = \sum_{k=0}^{\infty} 5(4)^k x^k - \sum_{k=0}^{\infty} 3(-2)^k x^k$$

$-\frac{1}{4} < x < \frac{1}{4}$        $-\frac{1}{2} < x < \frac{1}{2}$

} COMBINED WORKS FOR

$$= \sum_{k=0}^{\infty} (5 \cdot 4^k - 3(-2)^k) x^k$$

$$= (5 \cdot 4^0 - 3(-2)^0) x^0 + (5 \cdot 4 - 3(-2)) x^1 + (5 \cdot 16 - 3 \cdot 4) x^2 + \dots$$

$$= 2 + (20 + 6) x + (80 - 12) x^2 + \dots$$

$$= 2 + 26x + 68x^2 + \dots$$

(c)  $y = \cos^2(x)$  (Hint: Half-angle)

$$\begin{aligned}\cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \\ &= \frac{1}{2} + \frac{1}{2}\cos(2x)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k} \\ &= \boxed{\frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2(2k)!} 2^{2k} x^{2k}}\end{aligned}$$

ONLY CHANGES CONSTANT TERM

$$= \underbrace{\frac{1}{2} + \frac{1}{2}}_{k=0} - \underbrace{\frac{1}{2(2)!} 2^2 x^2 + \frac{1}{2(4)!} 2^4 x^4 - \dots}_{k=1, 2, \dots}$$

↑ CAN LEAVE LIKE THIS  
OR WRITE LIKE THIS

↓

$$= \boxed{1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2(2k)!} 2^{2k} x^{2k}}$$

SAME

For All  $x$

$$-\infty < x < \infty$$

## Integrating Applications

- (a) Give the first three nonzero terms of the Taylor Series for

$$\int_0^x 7 + 3t^5 e^{2t} dt$$

$$\approx \int_0^x 7 + 3t^5 + 6t^6 + \dots dt = \int_0^x 7 + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} t^{k+5} dt$$

$$= 7t + \frac{3}{6}t^6 + \frac{6}{7}t^7 \Big|_0^x$$

$$= 7x + \frac{3}{6}x^6 + \frac{6}{7}x^7 + \dots$$

$$= 7t + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} \frac{1}{k+6} t^{k+6} \Big|_0^x$$

$$= 7x + \sum_{k=0}^{\infty} \frac{3(2)^k}{k! (k+6)} x^{k+6}$$

For All  $x$ .

(b) Find a Taylor series for:

$$A(x) = \int_0^x \frac{\sin(t)}{t} dt$$

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \dots$$

$$\Rightarrow \frac{\sin(t)}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} = 1 - \frac{1}{3!} t^2 + \frac{1}{5!} t^4 - \frac{1}{7!} t^6 + \dots$$

$$\Rightarrow \int_0^x \frac{\sin(t)}{t} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! (2k+1)} t^{2k+1} \Big|_0^x = t - \frac{1}{3!} \frac{1}{3} t^3 + \frac{1}{5!} \frac{1}{5} t^5 - \dots \Big|_0^x$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! (2k+1)} x^{2k+1}$$

For All  $x$ .